2010年　数IB　中間　解答

[1]

(1)

$$\begin{aligned} f^{(4x)}(1) &= \sin 1 = \alpha, & f^{(4x+1)}(1) &= \cos 1 = \beta \\ f^{(4x+2)}(1) &= -\sin 1 = -\alpha, & f^{(4x+3)}(1) &= -\cos 1 = -\beta \end{aligned}$$

よって、

$$\begin{aligned} f(x) &= f(1) + \frac{f^{(1)}(1)}{1!}(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(6)}(1)}{6!}(x-1)^{(6)} + \frac{f^{(7)}(\mathrm{C})}{7!}(x-1)^7 \\ \sin x &= \alpha + \frac{\beta}{1!}(x-1) - \frac{\alpha}{2!}(x-1)^2 - \frac{\beta}{3!}(x-1)^3 + \frac{\alpha}{4!}(x-1)^4 + \frac{\beta}{5!}(x-1)^5 \\ &\quad -\frac{\alpha}{6!}(x-1)^6 - \frac{\cos C}{7!}(x-1)^7 \end{aligned}$$

[C は 1 と x の間の数]

(2)

$$f^{'}(\mathrm{x}) = \frac{1}{2\sqrt{1+\mathrm{x}}}, \quad f^{''}(\mathrm{x}) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{(1+\mathrm{x})^{\frac{3}{2}}}, \quad f^{3} = (-1)^2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{(1+\mathrm{x})^{\frac{5}{2}}}$$

$$f^{(4)}(\mathrm{x}) = (-1)^3\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{(1+\mathrm{x})^{\frac{7}{2}}}, \quad f^{n}(\mathrm{x}) = (-1)^{n-1}\cdot\frac{(2\mathrm{n}-3)!!}{2^n}\cdot\frac{1}{(1+\mathrm{x})^{\frac{2\mathrm{n}-1}{2}}}$$

$$\begin{aligned} \sqrt{1+x} &= 1 + \frac{1}{1}\cdot\frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{4}\right)x^2 + \frac{1}{3!}\cdot\frac{3}{8}x^3 + \frac{1}{4!}\cdot\left(-\frac{15}{16}\right)x^4 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots + \frac{(-1)^{n-2}(2n-5)!!}{(n-1)!2^{n-1}} + \frac{(-1)^{n-1}(2n-3)!!}{n!2^n}\cdot\frac{1}{(1+C)^{\frac{2n-1}{2}}} \end{aligned}$$

[C は 0 と x の間の数]

[2]

(1)

$$f(x,y) = x^3 - 3xy + y^3$$

$$\underline{f_x(x,\ y) = 3x^2} \quad , \quad \underline{f_y(x,\ y) = -3x + 3y^2}$$

(2)

$$f(x,\ y) = \sin\left(\frac{x}{y}\right) + \sin(xy)$$

$$\begin{aligned} \underline{f_x(x,\ y)} &= \cos\left(\frac{x}{y}\right)\cdot\frac{1}{y} + \cos(xy)\cdot y &&= \underline{\frac{1}{y}\cos\left(\frac{x}{y}\right) + y\cos(xy)} \\ \underline{f_y(x,\ y)} &= \cos\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right) + \cos(xy)\cdot x &&= \underline{-\frac{x}{y^2}\cos\left(\frac{x}{y}\right) + x\cos(xy)} \end{aligned}$$

(3)

$$\begin{aligned} f(x,\ y) &= x^y - y^x \\ f_x(x,\ y) &= yx^{y-1} - y^x\log y \quad , \quad f_y(x,\ y) = x^y\log x - xy^{x-1} \end{aligned}$$

[**3**]

(1)

$$\begin{aligned} f(1,\ 2) &= 1+8 = 9 \\ f_x(x,\ y) &= 4x^3 + 2y^2 \quad , \quad f_x(1,\ 2) = 4+8 = 12 \\ f_y(x,\ y) &= 4y \quad , \quad f_y(1,\ 2) = 8 \end{aligned}$$

$$z - 9 = 12(x-1) + 8(y-2) \qquad \underline{z = 12x + 8y - 19}$$

(2)

$$f\left(\frac{\pi}{6},\ \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f_x(x,\ y) = \cos(x+2y) \quad , \quad f_x\left(\frac{\pi}{6},\ \frac{\pi}{12}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$f_y(x,\ y) = 2\cos(x+2y) \quad , \quad f_y\left(\frac{\pi}{6},\ \frac{\pi}{12}\right) = 2\cos\frac{\pi}{3} = 1$$

$$z - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{6}\right) + \left(y - \frac{\pi}{3}\right)$$

$$z = \frac{1}{2}x + y - \frac{5}{12}\pi + \frac{\sqrt{3}}{2}$$

[4]

(1)

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} &&= \frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} &&= \frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}\cdot r\cos\theta \\
& &&= r\left(-\frac{\partial z}{\partial x}\cdot\sin\theta+\frac{\partial z}{\partial y}\cdot\cos\theta\right)
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta\right) \\
&= \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial r}\right\}\cos\theta+\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial r}\right\}\sin\theta \\
&= \left\{\frac{\partial^2 z}{\partial x^2}\cdot\cos\theta+\frac{\partial^2 z}{\partial x\partial y}\cdot\sin\theta\right\}\cos\theta+\left\{\frac{\partial^2 z}{\partial x\partial y}\cdot\cos\theta+\frac{\partial^2 z}{\partial y^2}\cdot\sin\theta\right\}\sin\theta \\
&= \frac{\partial^2 z}{\partial x^2}\cdot\cos\theta+2\cdot\frac{\partial^2 z}{\partial x\partial y}\cdot\sin\theta\cos\theta+\frac{\partial^2 z}{\partial y^2}\cdot\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left\{r\left(-\frac{\partial z}{\partial x}\cdot\sin\theta+\frac{\partial z}{\partial y}\cdot\cos\theta\right)\right\} \\
&= \left\{\frac{\partial}{\partial x}\cdot\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\cdot\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial \theta}\right\}(-r\sin\theta)+\left(-r\cdot\frac{\partial z}{\partial x}\right)\cdot\frac{\partial}{\partial \theta}(\sin\theta) \\
&\quad+\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial \theta}\right\}(r\cos\theta)+r\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial \theta}(\cos\theta) \\
&= \left\{\left(\frac{\partial^2 z}{\partial x^2}\right)\cdot(-r\sin\theta)+\left(\frac{\partial^2 z}{\partial x\partial y}\right)(r\cos\theta)\right\}(-r\sin\theta)-r\cos\theta\cdot\frac{\partial z}{\partial x} \\
&\quad+\left\{\frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)+\left(\frac{\partial^2 z}{\partial y^2}\right)\cdot(r\cos\theta)\right\}(r\cos\theta)-r\sin\theta\cdot\frac{\partial z}{\partial y} \\
&= r^2\left(\frac{\partial^2 z}{\partial x^2}\cdot\sin^2\theta-2\cdot\frac{\partial^2 z}{\partial x\partial y}\cdot\sin\theta\cos\theta+\frac{\partial^2 z}{\partial y^2}\cdot\cos^2\theta\right)-r\left(\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta\right)
\end{aligned}$$

[5] $u = x + y$, $v = x - y$ とおくと，$f(x,\ y) = g(x+y) + h(x-y)$

$$\iff f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right) = g(u) + h(v)$$

で $f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right)$ を $\mathrm{F}(u,\ v)$ とかくことにする。

$$\frac{\partial \mathrm{F}}{\partial x} = \frac{\partial \mathrm{F}}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \mathrm{F}}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial \mathrm{F}}{\partial u} + \frac{\partial \mathrm{F}}{\partial v} \qquad \text{同様に} \quad \frac{\partial \mathrm{F}}{\partial y} = \frac{\partial \mathrm{F}}{\partial u} - \frac{\partial \mathrm{F}}{\partial v}$$

$$\frac{\partial^2 \mathrm{F}}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial \mathrm{F}}{\partial u} + \frac{\partial \mathrm{F}}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial \mathrm{F}}{\partial u} + \frac{\partial \mathrm{F}}{\partial v}\right)\cdot\frac{\partial v}{\partial x} = \frac{\partial^2 \mathrm{F}}{\partial u^2} + 2\cdot\frac{\partial^2 \mathrm{F}}{\partial u \partial v} + \frac{\partial^2 \mathrm{F}}{\partial v^2} \qquad (1)$$

$$\frac{\partial^2 \mathrm{F}}{\partial y^2} = \frac{\partial}{\partial u}\left(\frac{\partial \mathrm{F}}{\partial u} - \frac{\partial \mathrm{F}}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial \mathrm{F}}{\partial u} - \frac{\partial \mathrm{F}}{\partial v}\right)\cdot\frac{\partial v}{\partial y} = \frac{\partial^2 \mathrm{F}}{\partial u^2} - 2\cdot\frac{\partial^2 \mathrm{F}}{\partial u \partial v} + \frac{\partial^2 \mathrm{F}}{\partial v^2} \qquad (2)$$

① = ②より，題意の必要十分条件は，

$$\frac{\partial^2 \mathrm{F}}{\partial u \partial v} = 0 \iff f\left(\frac{u+v}{2},\ \frac{u-v}{2}\right) = g(u) + h(v)$$

すなわち， $\underline{f(x,\ y) = g(x+y) + h(x-y)}$

[6]

(1)

$$^\forall\varepsilon > 0,\ ^\exists\delta > 0 \quad \text{s.t.} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,\ y) - f(a,\ b)| < \varepsilon$$

(2)

$$f(r\cos\theta,\ r\sin\theta) = \frac{r\cos\theta \cdot r\sin\theta}{\sin\left(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\right)} = \frac{r^2\sin\theta\cos\theta}{\sin|r|}$$

(3)

$$|f(r\cos\theta,\ r\sin\theta)| \leq \frac{|r|}{\sin|r|}\cdot|r|\cdot|\sin\theta\cos\theta| \leq \frac{|r|}{\sin|r|}\cdot|r| \longrightarrow 0$$

よって，$f(x,\ y)$ は $(0,\ 0)$ で連続。