証明

Step 1　2変数のテイラーの定理より，

$$f(a+s,\ b+t) - f(a,b) = \boxed{\left(s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\right) f(a,b)}$$

仮定より $=0$

$$+ \frac{1}{2}\left(s\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}\right)^2 f(a+\theta s,\ b+\theta t) \quad (0<\theta<1)$$

$$\text{第2項} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a+\theta s,\ b+\theta t)\cdot s^2 + \frac{\partial^2 f}{\partial x \partial y}(a+\theta s,\ b+\theta t)\cdot st + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a+\theta s,\ b+\theta t)\cdot t^2$$

$G(s,t)$ を次で定める

$(s,t)=(0,0)$ の時，$G(s,t)=0$ とおく．

$(s,t)\neq(0,0)$ の時

$$G(s,t) = \frac{s^2}{2(s^2+t^2)}\left\{\frac{\partial^2 f}{\partial x^2}(a+\theta s,\ b+\theta t) - \frac{\partial^2 f}{\partial x^2}(a,b)\right\}$$

$$+ \frac{st}{s^2+t^2}\left\{\frac{\partial^2 f}{\partial x \partial y}(a+\theta s,\ b+\theta t) - \frac{\partial^2 f}{\partial x \partial y}(a,b)\right\}$$

$$+ \frac{t^2}{2(s^2+t^2)}\left\{\frac{\partial^2 f}{\partial y^2}(a+\theta s,\ b+\theta t) - \frac{\partial^2 f}{\partial y^2}(a,b)\right\}$$

$\rightarrow$ $f(x,y)$ が C^2 級

$\Rightarrow$ $G(s,t)$ は $(s,t)=(0,0)$ で連続 で $G(s,t)=0$

かつ

$$f(a+s,\ b+t) - f(a,b) = \underbrace{\frac{1}{2}(As^2+2Bst+Ct^2)}_{\text{主要項}} + \underbrace{(s^2+t^2)G(s,t)}_{\text{おつり}}$$

無視できる

を みたす。㊟

但し，

$$A = \frac{\partial^2 f}{\partial x^2}(a,b)$$

$$B = \frac{\partial^2 f}{\partial x \partial y}(a,b) \qquad \left(\Leftrightarrow H_f(a,b) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}\right)$$

$$C = \frac{\partial^2 f}{\partial y^2}(a,b)$$